

Quantum mechanics and the Continuum Problem (II)

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Abstract

In one-dimensional case, it is shown that the basic principles of quantum mechanics are properties of the set of intermediate cardinality.

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The concept of discrete space is not a unique alternative of the continuous space. Since discrete space is a countable set, there is an intermediate possibility connected with the continuum problem: space may be neither continuous nor discrete. The commonly held view is that the independence of the continuum hypothesis (CH) is not a certain solution of the continuum problem in consequence of incompleteness of set theory. Nevertheless, from the independence of CH follows a unique definite status of the set of intermediate cardinality. It is important here that this set must be a subset of continuum (continuum must contain a subset equivalent to the intermediate set). Taking into account that any separation of the subset is a proof of existence of the intermediate set, which contradicts the independence of CH, we get that the set of intermediate cardinality exists only as a subset of continuum. In other words, the subset of intermediate cardinality, in principle, cannot be separated from continuum (set theory “confinement”). If Zermelo-Fraenkel set theory is consistent, complete, and giving the correct description of the notion of set, then this is the only possible understanding of the independence of CH.

Note that if we postulate existence of the intermediate set (in other words, if we take the negation of CH as an axiom), the result will be the

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same: since any construction or separation of the set are forbidden by the independence of CH, we have to reconcile with the same “latent” intermediate subset in continuum which we can get without any additional assumption. And it is not reasonable to take CH as an axiom because, as a consequence, we lose this subset.

According to the separation axiom schema, for any set X and for any property expressed by formula φ there exists a subset of the set X , which contains only members of X having φ . Then some subset cannot be separated from continuum if each point of the subset does not have its own peculiar properties but only combines properties of the members of the countable set and continuum.

At first sight, this seems to be meaningless. But the content of the requirement coincides with the content of wave-particle duality: quantum particle combines properties of a wave (continuum) and a point-like particle (the countable set).

As an illustration, consider a brick road which consists of black bricks and white bricks. If we know (or suspect) that among them there are some bricks which have white top side and black bottom side (or vice versa), we, nevertheless, cannot find them. Based only on top view, the problem of separation (and even existence) of black-and-white bricks is undecidable. Each brick can be black-and-white with some probability. However, if we have top view and bottom view, we can find these bricks: each of them looks like a white brick on the one view and like a black brick on the other view (“black-white duality”).

In order to get information about the “invisible” set consider the maps of the intermediate set I to the sets of real numbers (R) and natural numbers (N).

Let the map $I \rightarrow N$ decompose I into the countable set of equivalent mutually disjoint infinite subsets: $\cup I_n = I$ ($n \in N$). Let I_n be called a unit set. All members of I_n have the same countable coordinate n .

Consider the map $I \rightarrow R$. Continuum R contains a subset M equivalent to I , i.e., there exists a bijection

$$f : I \rightarrow M \subset R. \quad (1)$$

This bijection reduces to a separation of the intermediate subset M from continuum. Since any separation procedure is a proof of existence of the intermediate set and, therefore, contradicts the independence of the continuum hypothesis, we, in principle, do not have a rule for assigning a definite real number to a point of the intermediate set. Hence, any bijection can

take a point of the intermediate set only to a random real number. If we do not have preferable real numbers, then we have the equiprobable mapping. This already conforms to the quantum free particle. In the general case, we have the probability $P(r)dr$ of finding a point $s \in I$ about r .

Thus the point of the intermediate set has two coordinates: a definite natural number and a random real number:

$$s : (n, r_{\text{random}}). \quad (2)$$

Only the natural number coordinate gives reliable information about the relative positions of the points of the set and the size of its interval. But the points of a unit set are indistinguishable. It is clear that the probability $P(r)$ depends on the natural number coordinate of the corresponding point. Note that the information about a point in the one-dimensional intermediate set is necessarily two-dimensional.

For two real numbers a and b the probability $P_{a \cup b} dr$ of finding s in the union of the neighborhoods $(dr)_a \cup (dr)_b$

$$P_{a \cup b} dr \neq [P(a) + P(b)] dr \quad (3)$$

because s corresponds to both (all) points at the same time (the events are not mutually exclusive). It is convenient to introduce a function $\psi(r)$ such that $P(r) = \mathcal{P}[\psi(r)]$ and $\psi_{a \cup b} = \psi(a) + \psi(b)$. The idea is to compute the non-additive probability from some additive object by a simple rule.

We have

$$P_{a \cup b} = \mathcal{P}(\psi_{a \cup b}) = \mathcal{P}[\psi(a) + \psi(b)] \neq \mathcal{P}[\psi(a)] + \mathcal{P}[\psi(b)], \quad (4)$$

i.e., the dependence $\mathcal{P}[\psi(r)]$ is non-linear. The simplest non-linear dependence is a square dependence:

$$\mathcal{P}[\psi(r)] = |\psi(r)|^2. \quad (5)$$

The probability $P(r)$ is not probability density because we cannot integrate it due to its non-additivity (an integral is a sum). The normalization condition means only that f is a bijection: we can find only one image of the point s in R . Actually, the concept of probability should be modified. An illustration in terms of the above brick road will make this clear: If we know the exact number N_{B-W} of the black-and-white bricks, we do not need to check all the bricks of perhaps infinite brick road. It is reasonable to stop checking when all this bricks are obtained and put

$$P_{B-W} = \frac{N_{B-W}}{N_{\text{checked}}}, \quad (6)$$

where P_{B-W} is the probability of finding a black-and-white brick, $N_{checked}$ is the exact (minimal) number of the bricks checked. Thus only $N_{checked}$ may vary in the different test runs (finding all the black-and-white bricks) and we have to use the average value.

The concept of probability for continuum may be modified in a similar way, since the point always may be found in a finite interval. We do not need to take into consideration remaining empty continuum.

But we shall not alter the concept of probability because it is not altered in quantum mechanics (although this results in infinite probabilities). The main purpose of this paper is to show that quantum mechanics describes the set of intermediate cardinality.

The function ψ , necessarily, depends on n : $\psi(r) \rightarrow \psi(n, r)$. Since n is accurate up to a constant (shift) and the function ψ is defined up to the factor $e^{i\text{const}}$, we have

$$\psi(n + \text{const}, r) = e^{i\text{const}} \psi(n, r). \quad (7)$$

Hence, the function ψ is of the following form:

$$\psi(r, n) = A(r) e^{2\pi i n}. \quad (8)$$

Thus the point of the intermediate set corresponds to the function Eq.(8) in continuum. We can specify the point by the function $\psi(n, r)$ before the mapping and by the random real number and the natural number when the mapping has performed. In other words, the function $\psi(n, r)$ may be regarded as the image of s in R between mappings.

Consider probability $P(a, b)$ of finding the point s at b after finding it at a . Let us use a continuous parameter t for correlation between continuous and countable coordinates of the point s (simultaneity) and in order to distinguish between the different mappings (events ordering):

$$r(t_a), n(t_a) \rightarrow \psi(t) \rightarrow r(t_b), n(t_b), \quad (9)$$

where $t_a < t < t_b$ and $\psi(t) = \psi[n(t), r(t)]$. For simplicity, we shall identify the parameter with time without further discussion. Note that we cannot use the direct dependence $n = n(r)$. Since $r = r(n)$ is a random number, the inverse function is meaningless.

Assume that s is a “observable” point, i.e., for each $t \in (t_a, t_b)$ there exists the image of the point in continuum R .

Partition interval (t_a, t_b) into k equal parts ε :

$$k\varepsilon = t_b - t_a,$$

$$\begin{aligned}
\varepsilon &= t_i - t_{i-1}, \\
t_a &= t_0, t_b = t_k, \\
a &= r(t_a) = r_0, b = r(t_k) = r_k.
\end{aligned} \tag{10}$$

The conditional probability of finding the point s at $r(t_i)$ after $r(t_{i-1})$ is given by

$$P(r_{i-1}, r_i) = \frac{P(r_i)}{P(r_{i-1})} \tag{11}$$

(between the points t_{i-1} and t_i , the continuous image of the point is out of control but the unmonitored zone will be reduced to zero by passage to the limit $\varepsilon \rightarrow 0$), i.e.,

$$P(r_{i-1}, r_i) = \left| \frac{A_i}{A_{i-1}} e^{2\pi i \Delta n_i} \right|^2, \tag{12}$$

where $\Delta n_i = |n(t_i) - n(t_{i-1})|$. Note that Δn_i is really a vector.

The probability of the sequence of the transitions (we may use the word “transition” because we have the substantiated notion of time)

$$r_0, \dots, r_i, \dots r_k \tag{13}$$

is given by

$$P(r_0, \dots, r_i, \dots r_k) = P(r_1, r_2) \cdots P(r_{i-1}, r_i) \cdots P(r_{k-1}, r_k), \tag{14}$$

i.e.,

$$P(r_0, \dots, r_i, \dots r_k) = \left| \frac{A_k}{A_0} \exp 2\pi i \sum_{i=1}^k \Delta n_i \right|^2. \tag{15}$$

Then probability of the corresponding continuous sequence of the transitions $r(t)$

$$P[r(t)] = \lim_{\varepsilon \rightarrow 0} P(r_0, \dots, r_i, \dots r_k) = \left| \frac{A_k}{A_0} e^{2\pi i m} \right|^2, \tag{16}$$

where

$$m = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^k \Delta n_i. \tag{17}$$

Since at any time $t_a < t < t_b$ the point s corresponds to all points of R , it also corresponds to all continuous random sequences of mappings $r(t)$ simultaneously (we emphasize that $r(t)$ is not necessarily a classical path).

Probability $P[r(t)]$ of finding the point at any time $t_a \leq t \leq t_b$ on $r(t)$ is non-additive too. Therefore, we introduce an additive functional $\phi[r(t)]$. In the same way as above, we get

$$P[r(t)] = |\phi[r(t)]|^2. \quad (18)$$

Taking into account Eq.(16), we can put

$$\phi[r(t)] = \frac{A_N}{A_0} e^{2\pi i m} = \text{const } e^{2\pi i m}. \quad (19)$$

Thus we have

$$P(a, b) = \left| \sum_{\text{all } r(t)} \text{const } e^{2\pi i m} \right|^2, \quad (20)$$

i.e., the probability $P(a, b)$ of finding the point s at b after finding it at a satisfies the conditions of Feynman's approach (section 2-2 of [1]) for $S/\hbar = 2\pi m$ (indeed, Feynman does not essentially use in Chap. 2 that S/\hbar is just action).

Therefore,

$$P(a, b) = |K(a, b)|^2, \quad (21)$$

where $K(a, b)$ is path integral (2-25) of [1]:

$$K(a, b) = \int_{r_a}^{r_b} e^{2\pi i m} D r(t). \quad (22)$$

Thus we can apply Feynman's method in the following way.

1) We substitute $2\pi m$ for S/\hbar in in Eq.(2-15) of [1].

2) In section 2-3 of [1] Feynman explains how the principle of least action follows from the dependence

$$P(a, b) = \left| \sum_{\text{all } r(t)} \text{const } e^{(i/\hbar)S[r(t)]} \right|^2. \quad (23)$$

By the same nonrigorous reasoning, for "very, very" large m , we get "the principle of least m ". This also means that for large m the point s has a definite stationary path and, consequently, a definite continuous coordinate. In other words, the corresponding interval of the intermediate set is sufficiently close to continuum (let the interval be called macroscopic), i.e., cardinality of the intermediate set depends on its size. Recall that we can measure the size of an interval of the set only in the unit sets (some packets of points).

3) Since large m and Δn_i may be considered as continuous variables, we have

$$m = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^N \Delta n_i = \int_{t_a}^{t_b} dn(t) = \min. \quad (24)$$

The function $n(t)$ may be regarded as some function of $r(t)$: $n(t) = \eta[r(t)]$. It is important that $r(t)$ is not random due to the second item. Therefore,

$$\int_{t_a}^{t_b} dn(t) = \int_{t_a}^{t_b} \frac{d\eta}{dr} \dot{r} dt = \min, \quad (25)$$

where $\frac{d\eta}{dr} \dot{r}$ is some function of r , \dot{r} , and t (note absence of higher time derivatives than \dot{r}), i.e., large m can be identified with action:

$$m = \int_{t_a}^{t_b} L(r, \dot{r}, t) dt = \min. \quad (26)$$

Since the value of action depends on units of measurement, we need a parameter h (depending on units only) such that

$$hm = \int_{t_a}^{t_b} L(r, \dot{r}, t) dt. \quad (27)$$

Note that we can substitute action for m only for sufficiently high time rate of change of the countable coordinate n because, if $\Delta n_i = n(t_i) - n(t_{i-1})$ in Eq.(24) is not sufficiently large to be considered as an (even infinitesimal) interval of continuum, action reduces to zero. This may be understood as vanishing of mass of the point. Recall that mass is a factor which appear in Lagrangian of a free point as a peculiar property of the point under consideration, i.e., formally, mass may be regarded as a consequence of the principle of least action [2].

Finally, we may substitute S/\hbar for $2\pi m$ in Eq.(22) and apply Feynman's method to the set of intermediate cardinality.

Consider the special case of constant time rate of change ν of the countable coordinate n . We have $m = \nu(t_b - t_a)$. Then "the principle of least m " reduces to "the principle of least $t_b - t_a$ ". If ν is not sufficiently large (massless point), this is the simplest form of Fermat's least time principle for light. The more general form of Fermat's principle follows from Eq.(24): since

$$\int_{t_a}^{t_b} dn(t) = \nu \int_{t_a}^{t_b} dt = \min, \quad (28)$$

we obviously get

$$\int_{t_a}^{t_b} \frac{dr}{v(t)} = \min, \quad (29)$$

where $v(t) = dr/dt$. In the case of non-zero action (mass point), the principle of least action and Fermat's principle "work" simultaneously. It is clear that any additional factor can only increase the "pure least" time. As a result $t_b - t_a$ for a massless point bounds below $t_b - t_a$ for any other point and, therefore, $(b - a)/(t_b - t_a)$ for massless point bounds above average speed between the same points a and b for continuous image of any point of the intermediate set. This is a step towards special relativity.

It is important to make some general remarks on the description of the set intermediate cardinality.

The complete description of the intermediate set falls into two basic parts: continuous and countable. The continuous description is classical mechanics (the principle of least action is an intrinsic property of the set of intermediate cardinality).

Quantum mechanics is a connecting link and must be considered as a separate description (a countable description in terms of the continuous one). The description has its particular transitional main law (with action but without the principle of least action): the wave equation. Therefore, quantum mechanics is relevant for sufficiently large interval which may be considered as continuum. Compare this with the Copenhagen macroscopic measuring apparatus.

Thus the complete description of the intermediate set consist of three parts: macroscopic (continuous), microscopic in macroscopic terms (let us call it "submicroscopic"), and proper microscopic, i. e., it is a system of three dual theories.

Mathematical "invisibility" of the intermediate set leads to confusion: all descriptions are placed in the same continuous space. As a result the directions of the countable descriptions are lost and replaced with spin. We also lose microscopic dimensions of non-continuous descriptions.

The total number of space time dimensions of three 3D descriptions is ten. The same number of dimensions appear in string theories. But the extra dimensions of the intermediate set are essentially microscopic and do not require compactification. Since microscopic intervals (unlike macroscopic ones) are essentially non-equivalent, the proper microscopic description must split into a system of countable (quantum) dual "theories" with number of extra dimensions corresponding to the number of distinguishable cardinalities.

By definition, a proper microscopic interval can not be considered as continuous, i.e., it has no length. In other words, its macroscopic (continuous) image is exactly a point. Thus from macroscopic point of view, there are two kinds of points: the true points and the composite points. A composite

point consist of an infinite number of points. It is uniquely determined by the number of unit sets. Note that, in string theories, in order to get one natural number (mode) one needs at least two real numbers (length, tension) and additional assumptions. Cardinality of the proper microscopic interval may be regarded as some qualitative property of the point. This property vanishes if the interval is destroyed (decay of the corresponding point). The minimal building block for a composite point is a unit set. In the three-dimensional case, there must be three types of the unit sets forming, in the macroscopic limit, three-dimensional approximately continuous space.

References

- [1] Feynman R. P., Hibbs A. R., Quantum mechanics and Path Integrals, McGraw-Hill Book Company, New York, 1965
- [2] Landau L. D., Lifshitz E. M., Mechanics, Oxford; New York: Pergamon Press, 1976.